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Tutorial:

Bilevel Optimization Without Tears

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NEO
NUMERICAL AND EVOLUTIONARY OPTIMIZATION

Outline

Introduction
Optimization and origin of Bilevel Optimization (BO)

Bilevel Optimization
Practical examples to understand the formal definition and its properties

Solution Strategies
Methods from Classical to Metaheuristics

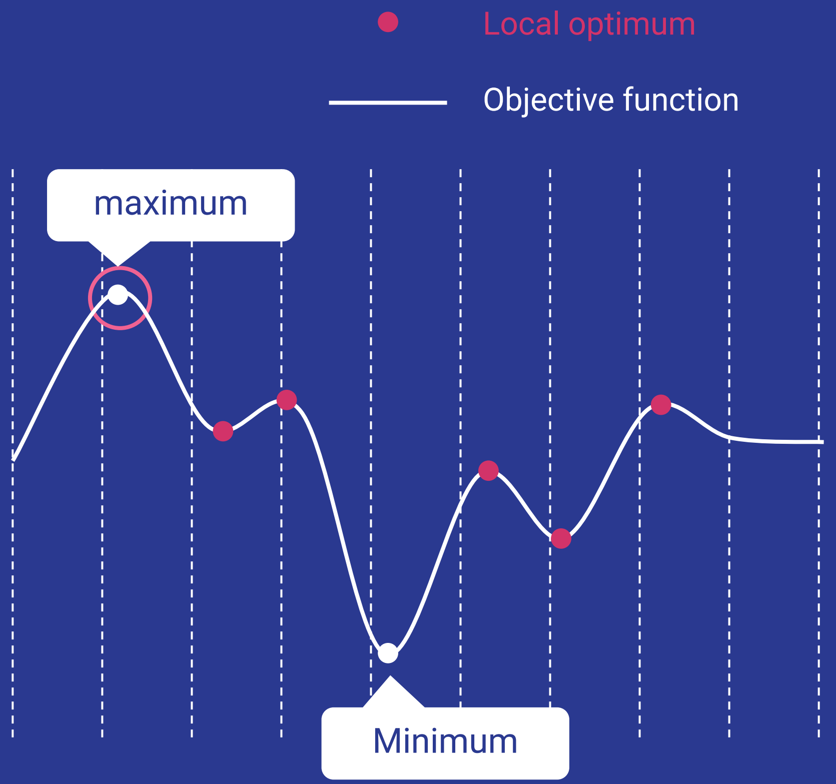
Applications
BO for solving interest problems

Conclusions
Final comments

Introduction

Optimization

101



Rao, S. S. (2019). Engineering optimization: theory and practice. John Wiley & Sons.

$$x^* \in \arg\min_{x \in X} f(x)$$

Optimization

101

Minimization

"Find a best element (with regard to some criterion) from a set of available alternatives"

Best element

Criterion

Set of alternatives

$$x^* \in \arg\min_{x \in X} f(x)$$

$$x^* \in \arg\min_{x \in X} f(x)$$

Optimization

101

Minimization

"Find an argument that minimizes an objective function"

Optimum

Objective function

$$x^* \in \arg\min_{x \in X} f(x)$$

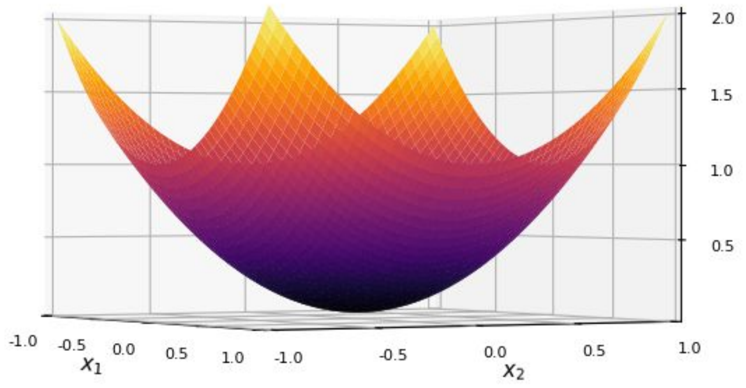
Search space

$$x^* \in \arg\min_{x \in X} f(x)$$

Optimization

101

Minimization

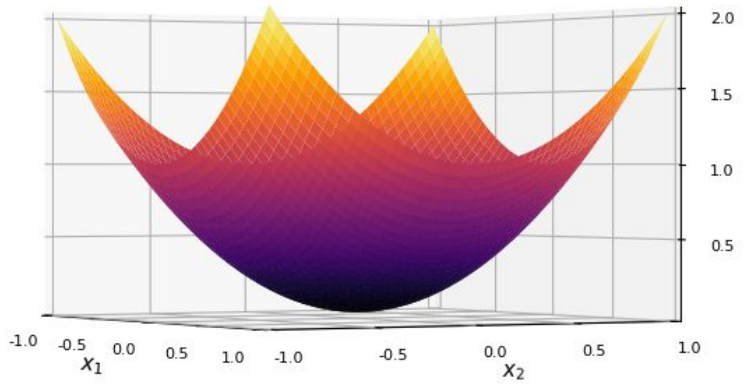


$$x^* \in \arg\min_{x \in X} f(x)$$

Optimization

Example

Minimization



$$f(x) = x_1^2 + x_2^2 \qquad X = [-1, 1]^2$$

$$x^* = (0, 0)$$

Optimization

Homework

Minimize

$$f(x) = \sum_{i=1}^{D} (x_i - y)^2$$

where $X = [-1, 1]^D$

and $\boldsymbol{y}$ is a constant equal to $\mathbf{1}$

Optimization

Homework

Minimize

$$f(x) = \sum_{i=1}^{D} (x_i - y)^2$$

where $X = [-1, 1]^D$

and $\boldsymbol{y}$ is a constant equal to $\mathbf{1}$

Solution:

$$x_i = y,\ 1 \leq i \leq D$$

Origin

Of Bilevel Optimization

- **1934** A Leader-follower game was introduced
- **1973** Formulation for bilevel optimization
- **1977** Introduce term "Bilevel Programming"
- **1988** BO problems are NP-hard!
- **1992** BO problems are **strongly** NP-hard!
- **1994** First Genetic Algorithm for BO
- **2000-Present** High interest from EC community

Sinha, A. et al (2017). A review on bilevel optimization: from classical to evolutionary approaches and applications. *IEEE Transactions on Evolutionary Computation*.

Bilevel Optimization

Bilevel Optimization

Wrong ideas on BO

"I am expert in Multi-objective optimization and my trusted friend told me that BO is a bi-objective optimization"

"Bilevel optimization is like optimizing 2D functions"

"Nobody needs BO"

Bilevel Optimization

Description

“An upper level authority takes a decision subject to an optimal response from a lower level authority”

Zhang, G., et al (2015). Multi-level decision making. New York: Springer.

Bilevel Optimization

Dummy example

"A guy finds ***the best way to escape****, subject to the* ***optimal path*** *(to the guy's position) planned by the chickens"*

Bilevel Optimization

Dummy example

*“A guy finds **the best way to escape**, subject to the **optimal path** (to the guy’s position) planned by the chickens”*

Best **leader** solution

Objective function

Best **follower** solution

$$x^* \in \arg\min_{x \in X} F(x, y^*)$$

Possible solutions

Bilevel Optimization

Dummy example

*"A guy finds **the best way to escape**, subject to the **optimal path** (to the guy's position) planned by the chickens"*

Best **follower** solution

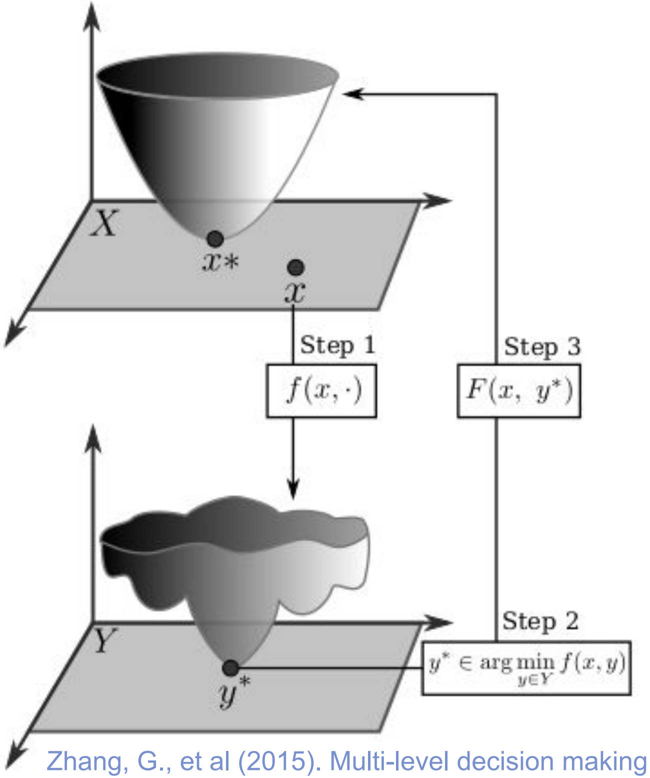
Objective function

Current **leader** solution

$$y^* \in \arg\min_{y \in Y} f(x, y)$$

Possible solutions

Bilevel Optimization: Nested Scheme



1. **Leader** takes a decision x
2. The **follower** uses the leader's decision to take the best decision based on f
3. The **leader** evaluates both x, y to evaluate F

Zhang, G., et al (2015). Multi-level decision making. New York: Springer.

Bilevel Optimization: Formal Definition

Minimize:

$$F(x, y^*),\ x \in X$$

Leader /
Upper Level

Subject to:

$$y^* \in \arg\min_{y \in Y} f(x, y)$$

Follower /
Lower Level

Dempe, Stephan. Foundations of bilevel programming. Springer Science & Business Media, 2002.

Bilevel Optimization: Solutions

$$(x, y^*)$$

Feasible solutions

$$x \in X$$

$$(x^*, y^*)$$

Optimal Feasible solutions

$$x^* \in \underset{x \in X, y^* \in \Psi(x)}{\arg\min} \; F(x.y^*)$$

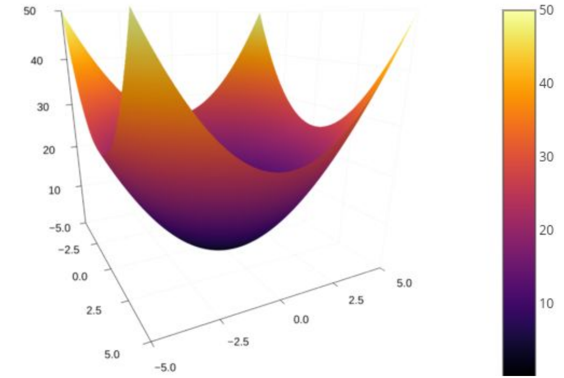
$$y^* \in \Psi(x) = \underset{y \in Y}{\arg\min} f(x, y)$$

Dempe, Stephan. Foundations of bilevel programming. Springer Science & Business Media, 2002.

Bilevel Optimization: Practical Example

Minimize:

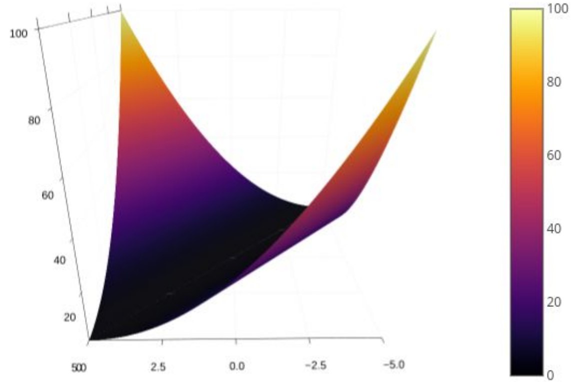
$$F(x, y) = x^2 + y^2$$



Subject to:

$$y \in \arg\min_{y \in Y} f(x, y) = (x - y)^2$$

$$X = Y = [-5, 5]$$



Bilevel Optimization: Practical Example

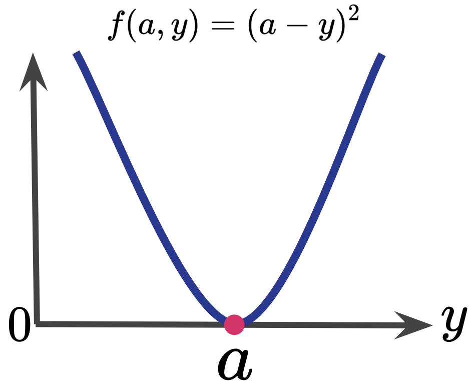
Step 1: Choose $x \in X$ said $x = a$

Step 2: Solve:

$$y \in \arg\min_{y \in Y} f(a, y) = (a - y)^2$$

$$f(a, y) \geq 0 \Rightarrow \min_{y} f(a, y) = 0 \Rightarrow y = a$$

$$\Psi(x) = \{a\}$$



Step 3: Evaluate: $F(a, y^*) = F(a, a) = 2a^2$

Bilevel Optimization: Practical Example

UL & LL functions

$$F(x, y) = x^2 + y^2$$

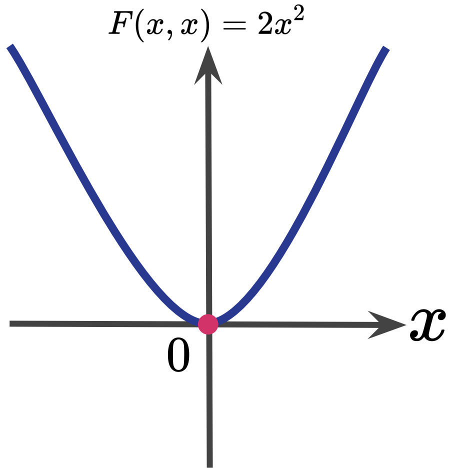
$$f(x, y) = (x - y)^2$$

Leader performance
s.t. best follower decision

$$F(x, y^*) = 2x^2$$

$$y^* = x$$

Best follower decision



Feasible solutions

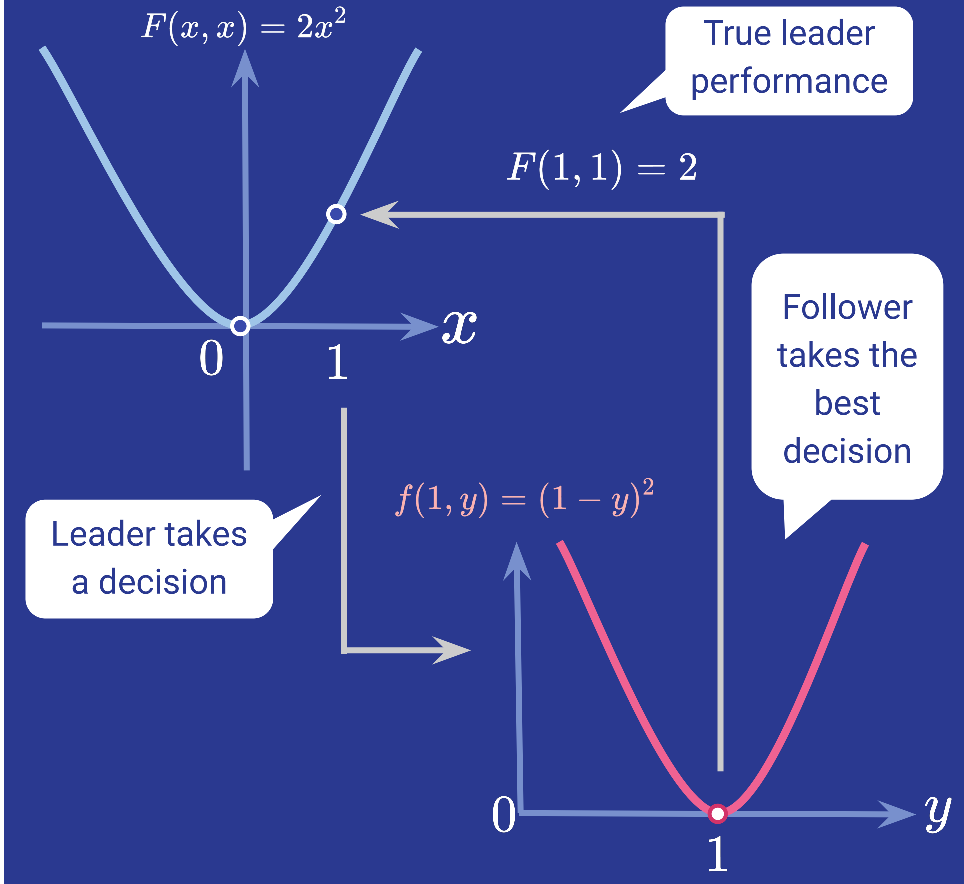
$$(x, x), -5 \leq x \leq 5$$

Optimum solution

$$(x^*, y^*) = (0, 0)$$

Bilevel Optimization

Practical Example



Bilevel Optimization: General Definition

Minimize:

$$F(x, y^*),\ x \in X$$

Subject to:

$$y^* \in \arg\min_{y \in Y}\{f(x,y) : g_i(x,y) \leq 0, i = 1,2,\ldots,I\}$$

$$G_j(x,y) \leq 0, j = 1,2,\ldots,J$$

Dempe, Stephan. Foundations of bilevel programming. Springer Science & Business Media, 2002.
Dempe, S., et al. (2015). Bilevel programming problems. Energy Systems. Springer, Berlin.

Solution Strategies

Solution Strategies

Classical

Mathematical programming

- Karush-Kuhn-Tucker conditions for single-level reduction
- Branch and Bound
- Trust region
- Among others

Approximate

Metaheuristics

Population-based algorithms have been successfully used.

- Evolutionary Algorithms
- Swarm Intelligence
- Among others

Hybrid

Metaheuristics + Mathematical programming

Usually, Karush-Kuhn-Tucker conditions are used in addition to population based algorithms for global optimization

Sinha, A. et al (2017). A review on bilevel optimization: from classical to evolutionary approaches and applications. *IEEE Transactions on Evolutionary Computation*.

Solution Strategies

Classical

Mathematical programming

- Karush-Kuhn-Tucker conditions for single-level reduction
- Branch and Bound
- Trust region
- Among others

Karush-Kuhn-Tucker conditions

$$\min_{x \in Y, y \in Y, \lambda} F(x, y)$$

Subject to:

$$\begin{aligned} G_j(x, y) &\leq 0, j = 1, 2, \ldots, J \\ g_i(x, y) &\leq 0, i = 1, 2, \ldots, I \\ \lambda_i g_i(x, y) &= 0, i = 1, 2, \ldots, I \\ \nabla_y L(x, y, \lambda) &= 0 \\ \lambda_i &\leq 0, i = 1, 2, \ldots, I \end{aligned}$$

$$L(x, y, \lambda) = f(x, y) + \sum_{i=1}^{I} \lambda_i g_i(x, y)$$

Solution Strategies

Approximate

Metaheuristics

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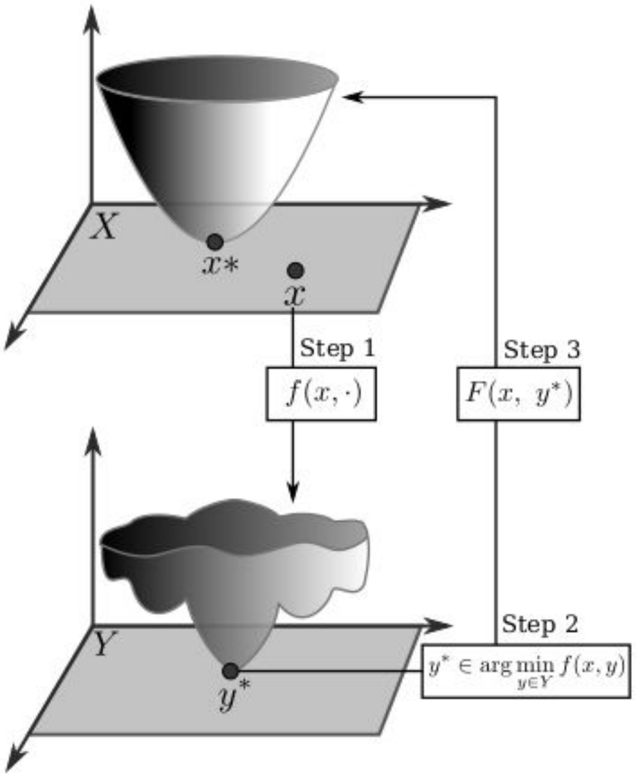
- Swarm Intelligence
- Among others



The metaheuristics use the nested scheme to sequentially optimize both levels.

Talbi, E. G. (2013). Metaheuristics for bi-level optimization. Springer, Berlin, Heidelberg.

Solution Strategies

Approximate

Metaheuristics

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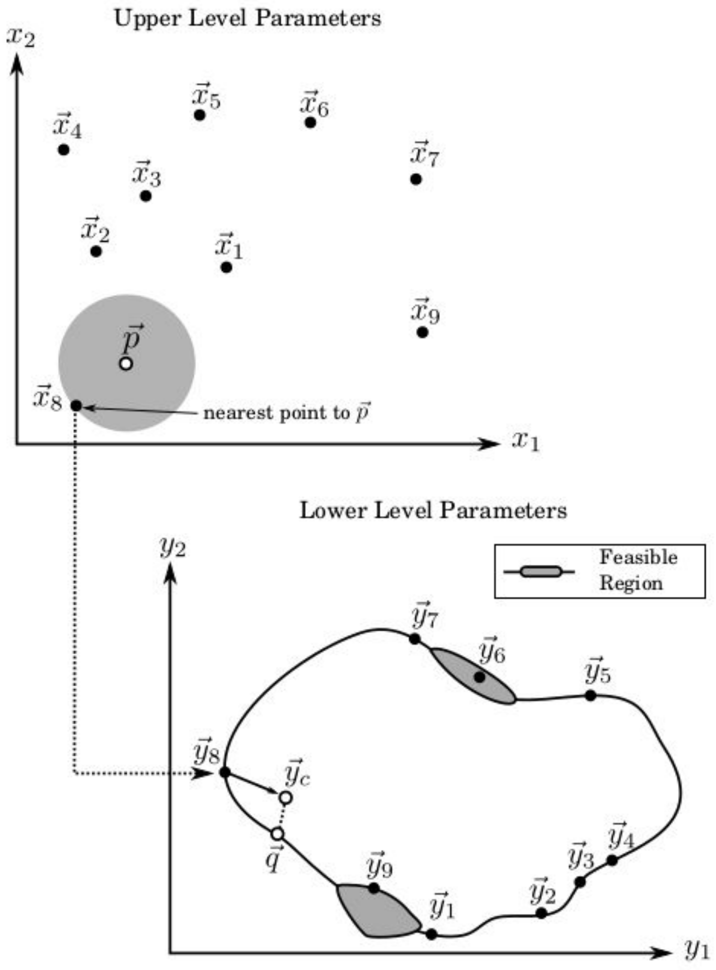

UL Population

LL Population

Talbi, E. G. (2013). Metaheuristics for bi-level optimization. Springer, Berlin, Heidelberg.

Solution Strategies

Approximate

Metaheuristics

Population-based algorithms have been successfully used.

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- Swarm Intelligence
- Among others

	Algorithms	
Paper	UL	LL
BLDES [2]	DE	DE
ε-KKT [38]	GA	GA/SQP
BLEGO [15]	DE/EGO	EGO
BLMA [16]	DE	DE
SABLA [14]	DE-IP	DE-IP
BLEAQ(-II) [32, 34]	GA	GA/SQP
Surrogate-assisted BIDE [1]	DE	DE

Differential Evolution (DE), Genetic Algorithm (GA), Sequential Quadratic Programming (SQP), Interior Point Method (IP), Efficient Global Optimization (EGO).

Mejía-de-Dios, J. A., & Mezura-Montes, E. (2020). A surrogate-assisted metaheuristic for bilevel optimization. GECCO 2020.

Solution Strategies

Simple (naive) **solution in Julia**

```
using Metaheuristics

F(x, y) = sum( x.^2 + y.^2 )

f(x,y) = sum( (x - y).^2 )

bounds = [ -5 -5 -5; 5 5 5.0 ]

Ψ(x) = minimizer( optimize( y -> f(x, y), bounds)  )

minimizer(optimize( x->F(x, Ψ(x)), bounds ) )
```

Solution found in 4.62 seconds.

```
▸ Float64[-0.0016458,  -0.000558254,  0.0014492]
  minimizer(res)

5.601127187667976e-6
  minimum(res)
```

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Applications

Applications

Toll Setting Problem

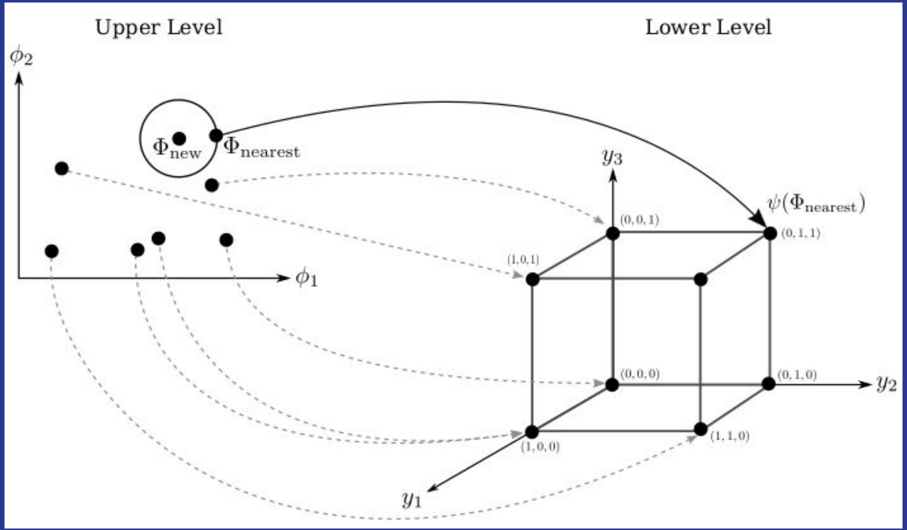
Upper level authority that wants to optimize the tolls revenue for a network of roads.

Followers are the network users that want to optimize their objectives (costs, time, etc).

Brotcorne, L., et al. (2001). A bilevel model for toll optimization on a multicommodity transportation network. Transportation science.

Applications

Automated Parameter Tuning



Leader: Chooses the parameters

Follower: Finds the hardest instances for the target algorithm

Working paper.

Applications

Principal-Agent Problems

Leader (principal) subcontracts a job to an agent (follower). Uses an incentive scheme that aligns the interests of the agent with the principal.

Follower: (agent) prefers to act in his own interests rather than those of the leader.

Cecchini, M., et al. (2013). Solving nonlinear principal-agent problems using bilevel programming. European Journal of Operational Research.

Conclusions

Conclusions

1. Bilevel Optimization (BO) is useful to model problem with inherent hierarchical structure
2. BO offers rich properties for solving problems
3. Approximate methods can successfully solve real-world problems
4. More theoretical studies are needed
5. BO problems can be hard to solve because of the computational complexity

Thank you!

Questions?

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